Complex numbers in standard form

Recall that the standard form of complex numbers is $a + bi$, where $a, b \in \mathbb{R}$

Exercise 1.1

Find the complex numbers in normal form corresponding to the following expressions:

a. $\left(\frac{1+i}{1-i}\right)^2$

b. $(1-i)(1+i)\frac{2}{2-i}$

c. $(-i)^{3253}$

d. $\frac{1-i^2+i^4-i^6+i^8-i^{10}}{1+i+i^2+i^3+i^4+i^5}$

e. $\sqrt{i}$

f. $\sqrt{-2i}$

g. $\sqrt{1+i\sqrt{3}} + \sqrt{1-i\sqrt{3}}$

Solution Exercise 1.1

a. $$\left(\frac{1+i}{1-i}\right)^2 = \frac{2i}{-2i} = \frac{-2}{2} = -1$$

b. $$(1-i)(1+i)\frac{2}{2-i} = \frac{4}{2-i} = \frac{8+4i}{5} = \frac{8}{5} + \frac{4}{5}i$$

c. because of the periodic nature of $(-i)^n = (-i)^{n+4}$ we can state $(-i)^n = (-i)^{n \bmod 4}$.

$(-i)^{3253 \bmod 4} = -i$

d. $$\frac{1-i^2+i^4-i^6+i^8-i^{10}}{1+i+i^2+i^3+i^4+i^5} = \frac{6}{1+i} = 3 - 3i$$

e. $$\sqrt{i} = \left(e^{\frac{1}{2}\pi i + k2\pi}\right)^{\frac{1}{2}} = e^{\frac{1}{4}\pi i + k\pi} = \pm\left(\cos\left(\frac{1}{4}\pi\right) + i\sin\left(\frac{1}{4}\pi\right)\right) = \pm\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i\right)$$

f. $$\sqrt{-2i} = i\sqrt{2}\sqrt{i} = i\sqrt{2}\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i\right) = -1 + i$$

g. $$\begin{aligned}\sqrt{1+i\sqrt{3}} + \sqrt{1-i\sqrt{3}} &= \sqrt{2}e^{\frac{1}{6}\pi i} + \sqrt{2}e^{-\frac{1}{6}\pi i} \\ &= \sqrt{2}\left(\frac{1}{2}\sqrt{3} + \frac{1}{2}i\right) + \sqrt{2}\left(\frac{1}{2}\sqrt{3} - \frac{1}{2}i\right) \\ &= \sqrt{6}\end{aligned}$$

Exercise 1.2

Prove the following properties for $z, w \in \mathbb{C}$

a. $\mathcal{Re}(z) = \mathcal{Im}(iz)$

b. $\mathcal{Im}(z) = \mathcal{Re}(-iz)$

c. $\bar{z} = 2\mathcal{Re}(z) - z$

d. $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$

Solution Exercise 1.2

let $z = a + bi$, $w = c + di$

a. $$\begin{aligned}\mathcal{Re}(z) &= \mathcal{Im}(iz)\\ \mathcal{Re}(a+bi) &= \mathcal{Im}(-b+ai)\\ a &= a\end{aligned}$$

b. $$\begin{aligned}\mathcal{Im}(z) &= \mathcal{Re}(-iz)\\ \mathcal{Im}(a+bi) &= \mathcal{Re}(b-ai)\\ b &= b\end{aligned}$$

c. $$\begin{aligned}\bar{z} &= 2\mathcal{Re}(z) - z\\ a - bi &= 2a - a - bi\\ a - bi &= a - bi\end{aligned}$$

d. $$\begin{aligned}|z+w|^2 + |z-w|^2 &= 2(|z|^2 + |w|^2)\\ (a+c)^2 + (b+d)^2 + (a-c)^2 + (b-d)^2 &= 2(a^2+b^2+c^2+d^2)\\ 2(a^2+c^2+b^2+d^2) &= 2(a^2+b^2+c^2+d^2)\end{aligned}$$

Exercise 1.3

Show that $\operatorname{card}(\mathbb{C}) = \operatorname{card}(\mathbb{R})$.

Solution Exercise 1.3

The steps of proofing this are: $\operatorname{card}([0,1]) = \operatorname{card}(\mathbb{R})$ then: $\operatorname{card}([0,1]) = \operatorname{card}([0,1]^2)$ followed by $\operatorname{card}([0,1]^2) = \operatorname{card}(\mathbb{R}^2)$ and finally $\operatorname{card}(\mathbb{R}^2) = \operatorname{card}(\mathbb{C})$

Complex numbers in polar form

Recall that the polar form of complex numbers is $r(\cos\theta + i\sin\theta)$ where $r \in \mathbb{R}_+$ and $\theta \in [0, 2\pi)$.

Exercise 2.4

Transform the following complex numbers from standard to polar form:

a. $-3 + 3i$
b. $-4\sqrt{3} - 4i$
c. $-5i$

Solution Exercise 2.4

a. $$\begin{aligned}-3+3i &= \sqrt{18}e^{\frac{3}{4}\pi i}\\ &= \sqrt{18}\left(\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right)\end{aligned}$$

b. $$\begin{aligned}-4\sqrt{3}-4i &= 8e^{\arctan_2\left(-4,-4\sqrt{3}\right)}\\ &= 8\left(\cos\left(-\frac{5}{6}\pi\right) + i\sin\left(-\frac{5}{6}\pi\right)\right)\end{aligned}$$

c. $-5i = 5e^{-\frac{1}{2}\pi}$
$$= 5\left(\cos\left(-\frac{1}{2}\pi\right) + i\sin\left(-\frac{1}{2}\pi\right)\right)$$

Exercise 2.5

Transform the following complex numbers from polar to standard form:

a. $2\left(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi\right)$

b. $3(\cos(-\pi) + i\sin(-\pi))$

c. $\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi$

Solution Exercise 2.5

a. $$2\left(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi\right) = 1 + \sqrt{3}i$$

b. $3(\cos(-\pi) + i\sin(-\pi)) = -3$

c. $$\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi = i$$

Exercise 2.6

Show that if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2\left(\cos\theta_{2_i}\sin\theta_2\right)$ are complex numbers in polar form, then:

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Solution Exercise 2.6

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{\theta_1 i}$

$z_2 = r_1(\cos\theta_2 + i\sin\theta_2) = r_2 e^{\theta_2 i}$

$$z_1 z_2 = r_1 e^{\theta_1 i} r_2 e^{\theta_2 i} = r_1 r_2 e^{(\theta_1 + \theta_2) i}$$
$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Exercise 2.7

Compute the following products by transforming the numbers to polar form:

a. $\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \cdot (-3 + 3i) \cdot \left(2\sqrt{3} + 2i\right)$

b. $(1 + i) \cdot (-2 - 2i) \cdot i$

Solution Exercise 2.7

a. $$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \cdot (-3+3i) \cdot \left(2\sqrt{3}+2i\right) = e^{-\frac{1}{3}\pi i} \cdot \sqrt{18}e^{\frac{3}{4}\pi i} \cdot 4e^{\frac{1}{6}\pi i}$$
$$= 12\sqrt{2}e^{\frac{7}{12}\pi i}$$

b. $$(1+i) \cdot (-2-2i) \cdot i = \sqrt{2}e^{\frac{1}{4}\pi i} \cdot \sqrt{8}e^{-\frac{3}{4}\pi i} \cdot e^{\frac{1}{2}\pi i}$$
$$= 4e^0$$
$$= 4$$

Exercise 2.8

Compute the following:

a. $(1+i)^{14}$

b. $(1 - \cos\alpha + i\sin\alpha)^n \quad \text{for } \alpha \in [0, 2\pi], n \in \mathbb{N}$

c. $z^n + \frac{1}{z^n}$ with $z + \frac{1}{z} = \sqrt{3}$

Solution Exercise 2.8

a. $$(1+i)^{14} = \left((1+i)^2\right)^7$$
$$= (1+2i-1)^7$$
$$= 128(i)^7$$
$$= -128i$$

b. Firstly write the tangent half angle formula as:

$$\tan\frac{1}{2}\alpha = \frac{\sin\alpha}{1+\cos\alpha} = -\frac{\sin(\alpha+\pi)}{1-\cos(\alpha+\pi)}$$

$$\Updownarrow$$

$$-\tan\left(\frac{1}{2}\alpha - \frac{1}{2}\pi\right) = \frac{\sin\alpha}{1-\cos\alpha}$$

We're going to write $(1-\cos\alpha + i\sin\alpha)^n$ in the form $re^{\varphi i}$. First we find φ:

$$\begin{aligned}\varphi &= \operatorname{arctan}_2(\sin\alpha, 1-\cos\alpha)\\ &= \arctan\left(\frac{\sin\alpha}{1-\cos\alpha}\right)\\ &= \arctan\left(-\tan\left(\frac{1}{2}\alpha - \frac{1}{2}\pi\right)\right)\\ &= -\arctan\left(\tan\left(\frac{1}{2}\alpha - \frac{1}{2}\pi\right)\right)\\ &= -\frac{1}{2}\alpha + \frac{1}{2}\pi \text{ for } \alpha \in [0, 2\pi]\end{aligned}$$

Next we find r:

$$\begin{aligned}r &= \sqrt{(1-\cos\alpha)^2 + \sin^2\alpha}\\ &= \sqrt{1 - 2\cos\alpha + \cos^2\alpha + \sin^2\alpha}\\ &= \sqrt{2-2\cos\alpha}\\ &= 2\left|\sin\left(\frac{1}{2}\alpha\right)\right| \quad \text{(This step isn't necessary, it helps computing the final answer)}\end{aligned}$$

Now we can solve the entire equation:

$$\begin{aligned}(1-\cos\alpha + i\sin\alpha)^n &= 2^n\left|\sin\frac{1}{2}\alpha\right|^n e^{-\frac{1}{2}\alpha n + \frac{1}{2}\pi n}\\ &= 2^n \sin^n\left(\frac{1}{2}\alpha\right) e^{\frac{1}{2}n(\pi-\alpha)}\end{aligned}$$

c. $z + \frac{1}{z} = \sqrt{3}$

$z^2 + 1 = \sqrt{3}z$

$z^2 - \sqrt{3} + 1 = 0$

$D = 3 - 4 = -1$

$z = \frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$

$z = e^{\frac{1}{6}\pi i} \vee z = e^{-\frac{1}{6}\pi i}$

$z^n + \frac{1}{z^n} = z^n + z^{-n}$

$e^{\frac{1}{6}\pi in} + e^{-\frac{1}{6}\pi in} \vee e^{-\frac{1}{6}\pi in} + e^{\frac{1}{6}\pi in}$ (this right side can be omitted as $\frac{1}{6} = -\frac{1}{6} \cdot -1$)

$z^n + \frac{1}{z^n} = e^{\frac{1}{6}\pi in} + e^{-\frac{1}{6}\pi in} = 2\cos\left(\frac{1}{6}\pi n\right)$

Complex roots of unity & polynomial equations

Exercise 3.9

Solve the following equations on $\mathbb{C}$

a. $z^2 = i$

b. $z^2 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

c. $z^3 + 2 - 2i = 0$

d. $z^3 + 4 - 4\sqrt{3}i = 0$

e. $z^4 = -7 + 24i$

f. $z^4 = -7 + 4\sqrt{2}i$

g. $z^8 = \sqrt{3} + i$

h. $z^7 - 2iz^4 - iz^3 - 2 = 0$

i. $z^6 + iz^3 + i - 1 = 0$

Solution Exercise 3.9

a. $z^2 = e^{\frac{1}{2}\pi i + k2\pi i}$

$z = e^{\frac{1}{4}\pi i + k\pi i}$

$z = \frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2} \vee z = z = -\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}$

b.

$$
\begin{aligned}
z^2 &= \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i \\
z^2 &= e^{\frac{1}{4}\pi i + k2\pi i} \\
z &= e^{\frac{1}{8}\pi i + k\pi i} \\
z &= \cos\frac{1}{8}\pi + i\sin\frac{1}{8}\pi \lor z = \cos\frac{9}{8}\pi + i\sin\frac{9}{8}\pi \\
z &= \frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}} \lor z = -\frac{1}{2}\sqrt{2+\sqrt{2}} - i\frac{1}{2}\sqrt{2-\sqrt{2}}
\end{aligned}
$$

note: the exact answers were found using the half angle formulae of sin and cos, there are many other ways to compute an exact answer void of trigonometric functions.

c.

$$
\begin{aligned}
z^3 &= -2 + 2i \\
z^3 &= 8^{\frac{1}{6}} e^{\frac{3}{4}\pi i + k2\pi i} \\
z &= \sqrt{2}e^{\frac{1}{4}\pi i + k\frac{2}{3}\pi i} \\
z &= 1 + i \lor z = -\frac{1}{2} + \frac{1}{2}\sqrt{3} - \left(\frac{1}{2} + \frac{1}{2}\sqrt{3}\right)i \lor z = -\frac{1}{2} - \frac{1}{2}\sqrt{3} - \left(\frac{1}{2} - \frac{1}{2}\sqrt{3}\right)i
\end{aligned}
$$

d.

$$
\begin{aligned}
z^3 &= -4 + 4\sqrt{3}i \\
z^3 &= 8e^{\operatorname{arctan}_2\left(4\sqrt{3},-4\right)i + k2\pi i} \\
z &= 2e^{\frac{2}{9}\pi i + k\frac{6}{9}\pi i} \\
z &= 2\cos\left(\frac{2}{9}\pi\right) + 2i\sin\left(\frac{2}{9}\pi\right) \\
\lor\ z &= 2\cos\left(\frac{8}{9}\pi\right) + 2i\sin\left(\frac{8}{9}\pi\right) \\
\lor\ z &= 2\cos\left(\frac{14}{9}\pi\right) + 2i\sin\left(\frac{14}{9}\pi\right)
\end{aligned}
$$

note that $\operatorname{arctan}_2\left(4\sqrt{3}, -4\right) = \frac{2}{3}\pi \neq \arctan\left(\frac{4\sqrt{3}}{-4}\right)$ *as* arctan_2 *returns the angle in the right quadrant.*

e. $z^4 = -7 + 24i$

because $\arctan_2(24, -7)$ does not have an exact solutions, we will use substitution

$z^2 = u = a + bi$ with $a, b \in \mathbb{R}$

$u^2 = -7 + 24i$

$a^2 + 2abi - b^2 = -7 + 24i$

$\Rightarrow a^2 - b^2 = -7 \land 2ab = 24$

$a = \frac{12}{b}$

$144b^{-2} - b^2 = -7$

$b^4 - 7b^2 - 144 = 0$

$(b^2 - 16)(b^2 + 9) = 0$

$b^2 = 16 \lor b^2 = -9$ (spurious)

$b = \pm 4$

$u = 3 + 4i \lor u = -3 - 4i$

$$x^2 = 3 + 4i \lor x^2 = -3 - 4i$$
$$a^2 + 2abi - b^2 = 3 + 4i \lor a^2 + 2abi - b^2 = -3 - 4i$$
$$(a^2 - b^2 = 3 \land 2ab = 4) \lor (a^2 - b^2 = -3 \land 2ab = -4)$$
$$a = \frac{2}{b} \lor a = -\frac{2}{b}$$
$$4b^{-2} - b^2 = 3 \lor 4b^{-2} - b^2 = -3$$
$$b^4 + 3b^2 - 4 = 0 \lor b^4 - 3b^2 - 4 = 0$$
$$(b^2 + 4)(b^2 - 1) = 0 \lor (b^2 - 4)(b^2 + 1) = 0$$
$$b^2 = -4 \text{ (spurious)} \lor b^2 = 1 \lor b^2 = 4 \lor b^2 = -1 \text{ (spurious)}$$
$$b = 1 \lor b = -1 \lor b = 2 \lor b = -2$$
$$x = 2 + i \lor x = -2 - i \lor x = -1 + 2i \lor x = 1 - 2i$$

f. $z^4 = -7 + 4\sqrt{2}i$

using substitution

$z^4 = u^2 = a + bi$ with $a, b \in \mathbb{R}$ $u^2 = -7 + 4\sqrt{2}i$

$a = \frac{2}{b}\sqrt{2}$

$b^4 - 7b^2 - 8 = 0$

$(b^2 - 8)(b^2 + 1)$

$b^2 = 8 \lor b^2 = -1$ (spurious)

$b = \pm 2\sqrt{2}$

$u = 1 + 2\sqrt{2}i \lor u = -1 - 2\sqrt{2}i$

$$x^2 = 1 + 2\sqrt{2}i \lor x^2 = -1 - 2$$

$$a = \frac{\sqrt{2}}{b} \lor a = -\frac{\sqrt{2}}{b}$$

$$b^4 + b^2 - 2 = 0 \lor b^4 - b^2 - 2 = 0$$

$$(b^2 + 2)(b^2 - 1) \lor (b^2 - 2)(b^2 + 1)$$

$$b^2 = -2 \text{ (spurious)} \lor b^2 = 1 \lor b^2 = 2 \lor b^2 = -1 \text{ (spurious)}$$

$$b = \pm 1 \lor b = \pm\sqrt{2}$$

$$x = \sqrt{2} + i \lor x = -\sqrt{2} - i \lor x = -1 + \sqrt{2}i \lor x = 1 - \sqrt{2}i$$

g. $z^8 = \sqrt{3} + i$

$z^8 = 2e^{\frac{1}{6}\pi + k2\pi}$

$z = 2^{\frac{1}{8}} e^{\frac{1}{48}\pi i + k\frac{12}{48}\pi i}$

$z = 2^{\frac{1}{8}} \cos\left(\frac{1}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{1}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{13}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{13}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{25}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{25}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{37}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{37}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{49}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{49}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{61}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{61}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{73}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{73}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{85}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{85}{48}\pi\right)$

$\lor z = 2^{\frac{1}{8}} \cos\left(\frac{97}{48}\pi\right) + 2^{\frac{1}{8}} i \sin\left(\frac{97}{48}\pi\right)$

h. $z^7 - 2iz^4 - iz^3 - 2 = 0$

$(z^4 - i)(z^3 - 2i) = 0$

$z^4 = i \vee z^3 = 2i$

$z^2 = \pm\sqrt{i} \vee z = 2^{\frac{1}{3}} e^{\frac{1}{6}\pi i + k\frac{4}{6}\pi i}$

$z = \frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}}$

$\vee\, z = -\frac{1}{2}\sqrt{2+\sqrt{2}} + -i\frac{1}{2}\sqrt{2-\sqrt{2}}$

$\vee\, z = \frac{1}{2}\sqrt{2+\sqrt{2}} - i\frac{1}{2}\sqrt{2-\sqrt{2}}$

$\vee\, z = -\frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}}$

$\vee\, z = -2^{-\frac{2}{3}}\sqrt{3} + 2^{-\frac{2}{3}} i$

$\vee\, z = 2^{-\frac{2}{3}}\sqrt{3} + 2^{-\frac{2}{3}} i$

$\vee\, z = -i2^{\frac{1}{3}}$

i. $x^6 + ix^3 + i - 1 = 0$

$x^3 = u = a + bi$ with $a, b \in \mathbb{R}$

$u^2 + iu + i - 1 = 0$

$(u + 1)(u + i - 1) = 0$

$u = -1 \vee u = 1 - i$

$x^3 = -1 \vee x^3 = 1 - i$

$x = -1$

$\vee\, x = \frac{1}{2} + i\frac{1}{2}\sqrt{3}$

$\vee\, x = \frac{1}{2} - i\frac{1}{2}\sqrt{3}$

$\vee\, x = 1 - i$

$\vee\, x = \sqrt{2}\cos\left(\frac{5}{12}\pi\right) + i\sqrt{2}\sin\left(\frac{5}{12}\pi\right)$

$\vee\, x = \sqrt{2}\cos\left(\frac{13}{12}\pi\right) + i\sqrt{2}\sin\left(\frac{13}{12}\pi\right)$

Exercise 3.10

Find all solutions to the equation $z^5 = 2 - 2i$, rounded to three digits.

Solution Exercise 3.10

$$
\begin{aligned}
z^5 &= 2 - 2i \\
z^5 &= \sqrt{8} e^{-\frac{3}{4}\pi i + k 2\pi i} \\
z &= 8^{\frac{1}{10}} e^{-\frac{3}{20}\pi i + k \frac{2}{5}\pi i} \\
z &= -0.870 + 0.870i \\
\lor\ z &= 0.192 - 1.216i \\
\lor\ z &= 0.559 + 1.097i \\
\lor\ z &= 1.216 - 0.192i \\
\lor\ z &= -1.097 - 0.559i
\end{aligned}
$$